JOURNAL OF DIFFERENTIAL EQUATIONS 43, 168–183 (1982)

Perturbed Volterra Equations

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Received October 31, 1980; revised April 17, 1981

1. INTRODUCTION

We consider a system of Volterra equations

$$x'(t) = Ax(t) + \int_0^t C(t,s)\,E(x(s))\,x(s)\,ds + F(t, x(s); 0 \leqslant s \leqslant t) \tag{1}$$

in which A is an $n \times n$ constant matrix all of whose characteristic roots have negative real parts, C is an $n \times n$ matrix of functions continuous for $0 \leqslant s \leqslant t < \infty$, E is an $n \times n$ matrix of functions continuous for all x in R^n, and F is a column vector functional which is continuous whenever $x(s)$ is a continuous function in R^n on $[0, t]$.

We shall suppose the functions well enough behaved that for each $t_0 \geqslant 0$ and each continuous function ϕ: $[0, t_0] \to R^n$, there is a solution $x(t)$ of (1) on $[t_0, t_0 + T)$ for some $T > 0$ and $x(t) = \phi(t)$ on $[0, t_0]$. We also assume that if a solution remains bounded, then it can be continued for all future time. The reader may consult Driver [5] for conditions ensuring such behavior and for information concerning stability and Liapunov's second method.

For motivation and clarity of exposition it will be convenient to also consider a linear scalar form of (1) with a simplified F which we write as

$$y'(t) = \mu(t)\,y(t) + \int_0^t h(t,s)\,y(s)\,ds + f(t) \tag{2}$$

with μ and f continuous scalar functions on $[0, \infty)$, while $h(t,s)$ is a continuous scalar function for $0 \leqslant s \leqslant t < \infty$.

In recent years several authors have investigated various forms of (1) and (2) using either Laplace transforms (cf. Brauer [1] or Miller [9]), Liapunov functions (cf. Grimmer and Seifert [6]), Liapunov functionals (cf. Burton [2, 3]), or Gronwall's inequality (cf. Burton [4]). In the linear convolution case we noted [2; p. 117] that the perturbation problem was the same as for ordinary differential equations, as the variation of parameters formulae are

0022-0396/82/020204-31$02.00/0

identical. But in the non-convolution or nonlinear case the term F (or f) is always clumsy to handle, and there seems to be no unified approach. One generally lists properties of F or f and handles them separately (cf. [6, 7]).

In this paper we construct Liapunov functionals $V(t, x(\cdot))$ with the property that

$$V'_{(1)}(t, x(\cdot)) \leqslant -bV(t, x(\cdot)) + K\,|F(t, x(\cdot))|, \tag{3}$$

where b and K are positive constants so that from (3) follows the familiar variation of parameters relation

$$V(t, x(\cdot)) \leqslant e^{-b(t-t_0)} V(t_0, \phi) + \int_{t_0}^{t} e^{-b(t-s)} K\,|F(s, x(s))|\,ds \tag{4}$$

from which a wealth of information may be deduced. In the construction it often turns out that V is not positive definite and, hence, boundedness and uniform boundedness result from (4), but not ultimate boundedness.

The functionals take four basic forms which we abbreviate here taking $E(x) = I$,

$$V_1(t, x(\cdot)) = [x^T Bx]^{1/2} + \int_0^t \int_t^\infty [\lambda(u)/\lambda(t)]\,|C(u, s)|\,du\,|x(s)|\,ds, \tag{5}$$

where $A^T B + BA = -I$ and $\lambda(t)$ is chosen so that $V'_{1(1)}(t, x(\cdot)) \leqslant -\gamma(t)\,V_1(t, x(\cdot)) + K\,|F(t, x(\cdot))|$ for a constant K and a non-negative function γ. This yields ultimate boundedness results.

$$V_2(t, x(\cdot)) = [x^T Bx]^{1/2} + \int_0^t \left[\alpha(s)\,e^{-b(t-s)} - \int_s^t K e^{-b(t-u)}\,|C(u, s)|\,du\right] |x(s)|\,ds \tag{6}$$

which results in (3) and which yields uniform boundedness and uniform stability properties.

$$V_3(t, x(\cdot)) = V_2(t, x(\cdot)) \qquad \text{with} \quad \alpha \equiv 0. \tag{7}$$

This form is strictly for boundedness, but it makes minimal demands on (1). Notice that V_3 takes on both positive and negative values.

$$V_4(t, x(\cdot)) = x^T Bx + \int_0^t x^T(s) \left[H(s)\,e^{-b(t-s)} + \int_s^t R(u, s)\,e^{-b(t-u)}\,du\right] x(s)\,ds \tag{8}$$

with H an $n \times n$ symmetric matrix and R a scalar matrix. It is very flexible, but it is not globally Lipschitz in $x(t)$ and, hence, is not so effective in perturbations. In particular, (3) will not hold.

The manner in which a non-positive functional may be used for boundedness is one of the main novelties of the paper.

2. Uniform Boundedness

We begin with (2) and construct a functional

$$V(t, y(\cdot)) = |y(t)| + \int_0^t \left[\alpha(s)\, e^{-b(t-s)} - \int_s^t e^{-b(t-u)}\, |h(u, s)|\, du\right] |y(s)|\, ds \tag{9}$$

in which α is a continuous scalar function and b is a positive constant.

THEOREM 1. *Suppose there is a continuous scalar function $\alpha\colon [0, \infty) \to R$ and a positive constant b such that*

$$\mu(t) + \alpha(t) \leqslant -b. \tag{10}$$

Then the functional V defined in (9) *satisfies*

$$V'_{(2)}(t, y(\cdot)) \leqslant -bV(t, y(\cdot)) + |f(t)| \tag{11}$$

so that

$$V(t, y(\cdot)) \leqslant V(t_0, \phi)\, e^{-b(t-t_0)} + \int_{t_0}^t e^{-b(t-s)}\, |f(s)|\, ds \tag{12}$$

along any solution $y(t)$ of (2).

Proof. We find that

$$\begin{aligned} V'_{(2)}(t, y(\cdot)) \leqslant{}& \mu(t)\, |y(t)| + \int_0^t |h(t, s)\, y(s)|\, ds + |f(t)| \\ &+ \alpha(t)\, |y(t)| + \int_0^t \left[-b\alpha(s)\, e^{-b(t-s)} \right. \\ &\left. - |h(t, s)| + \int_s^t be^{-b(t-u)}\, |h(u, s)|\, du\right] |y(s)|\, ds \end{aligned}$$

$$\leqslant [\mu(t)+\alpha(t)]\,|y|+|f(t)|-b\int_0^t \left[\alpha(s)\,e^{-b(t-s)}\right.$$

$$\left.-\int_s^t e^{-b(t-u)}\,|h(u,s)|\,du\right]|y(s)|\,ds$$

$$\leqslant -bV(t, y(\cdot))+|f(t)|$$

as required. The remainder follows immediately.

COROLLARY 1.1. *Let the conditions of Theorem 1 hold and suppose that there is a continuous scalar function $\Phi(t,s)\geqslant 0$ for $0\leqslant s\leqslant t<\infty$ with*

$$\alpha(s)\,e^{-b(t-s)}-\int_s^t e^{-b(t-u)}\,|h(u,s)|\,du\geqslant -\Phi(t,s) \tag{13}$$

for $0\leqslant s\leqslant t<\infty$ and that

$$\int_0^t \Phi(t,s)\,ds\leqslant P<1 \tag{14}$$

for some constant P and for $0\leqslant t<\infty$. Then any solution $y(t)$ of (2) on an interval $[t_0, T]$ having $|y(T)|$ as the maximum on $[0, T]$ satisfies

$$|y(T)|\leqslant \left[V(t_0,\phi)\,e^{-b(T-t_0)}\right.$$

$$\left.+\int_{t_0}^T e^{-b(T-s)}\,|f(s)|\,ds\right]\Big/[1-P]. \tag{15}$$

Proof. Notice that $y(t)=\phi(t)$ on $[0,t_0]$, but y is only a solution on $[t_0, T]$. We have $y(t)=\phi(t)$ on $[0,t_0]$ and we are asking that $|y(T)|\geqslant|y(t)|$ on $[0, T]$. Thus, the corollary only speaks about solutions which, in fact, achieve a maximum norm outside their initial interval.

Thus, let $y(t)$ be a solution of (2) on an interval past t_0 and suppose there is a $T\geqslant t_0$ with $|y(T)|\geqslant|y(t)|$ for $0\leqslant t\leqslant T$. With (12), (13), and (14) in mind we write

$$|y(T)|[1-P]=|y(T)|-P\,|y(T)|$$

$$\leqslant|y(T)|-\int_0^T \Phi(T,s)\,|y(T)|\,ds$$

$$\leqslant|y(T)|-\int_0^T \Phi(T,s)\,|y(s)|\,ds$$

$$\leq |y(T)| + \int_0^T \left[\alpha(s)\, e^{-b(T-s)} - \int_s^T e^{-b(T-u)}\, |h(u,s)|\, du\right] |y(s)|\, ds$$

$$= V(T, y(\cdot)) \leq V(t_0, \phi)\, e^{-b(T-t_0)} + \int_{t_0}^T e^{-b(T-s)}\, |f(s)|\, ds$$

from which the result follows.

Remark 1. Let us examine (13) and (14) with $\alpha(s) \equiv 0$. We then have

$$\begin{aligned}\int_0^t \Phi(t,s)\, ds &= \int_0^t \int_s^t e^{-b(t-u)}\, |h(u,s)|\, du\, ds\\ &= \int_0^t \int_0^u e^{-b(t-u)}\, |h(u,s)|\, ds\, du\\ &= \int_0^t e^{-b(t-u)} \int_0^u |h(u,s)|\, ds\, du.\end{aligned}$$

Thus, if there exists $b > 0$ with $\mu(t) \leq -b$ and if there exists $\bar{b} < b$ with $\int_0^u |h(u,s)|\, ds \leq \bar{b}$, then

$$\int_0^t \Phi(t,s)\, ds \leq \bar{b} \int_0^t e^{-b(t-u)}\, du \leq \frac{\bar{b}}{b} \stackrel{\text{def}}{=} P < 1$$

and (14) is satisfied.

COROLLARY 1.2. *Let the conditions of Corollary* 1.1 *hold, let* $\alpha(t) \leq M$, *and let* $\int_0^t e^{-b(t-s)}\, |f(s)|\, ds \leq M$ *for* $0 \leq t < \infty$ *and some* $M > 0$. *Then solutions of* (2) *are uniform bounded and, when* $f(t) \equiv 0$, *the zero solution of* (2) *is uniformly stable.*

Proof. Notice that for $\|\phi\| = \max_{0 \leq s \leq t_0} |\phi(s)|$ we have

$$V(t_0, \phi) \leq \|\phi\| \left[1 + \int_0^{t_0} M e^{-b(t_0 - s)}\, ds\right] \leq \|\phi\| [1 + (M/b)]$$

so that (15) will yield

$$|y(T)| \leq \{\|\phi\| [1 + (M/b)] + M\} / [1 - P]$$

from which the first conclusion follows. (That is, solutions are *uniform bounded* if for each $B_1 > 0$ there exists $B_2 > 0$ such that $\|\phi\| < B_1$ implies

$|y(t)| < B_2$ on $[t_0, \infty)$, independent of t_0.) The second conclusion follows, as the second M in the preceding bound on $|y(T)|$ is zero. (That is, the zero solution of (2) (when $f(t) \equiv 0$) is *uniformly stable* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\phi\| < \delta$ implies $|y(t)| < \varepsilon$ for $t \geqslant t_0$, independent of t_0.)

COROLLARY 1.3. *If the conditions of Corollary* 1.1 *hold, if* $\Phi(t, s) = 0$, *if* $\int_0^t e^{-b(t-s)} |f(s)|\, ds \leqslant M$, *and if* $\alpha(t) \leqslant M$ *for some* $M > 0$, *then solutions of* (2) *are uniformly ultimately bounded. If, in addition,* $f(t) \equiv 0$, *then* $y = 0$ *is exponentially asymptotically stable.*

Proof. (We say that solutions of (2) are *uniformly ultimately bounded* if they are uniform bounded and if there exists $L > 0$ such that for each $B_1 > 0$ there exists $T > 0$ so that $|\phi(t)| < B_1$ on $[0, t_0]$ implies $|x(t, \phi)| < L$ if $t \geqslant t_0 + T$, independent of t_0.) If $\Phi(t, s) = 0$, then from (9) and (13) we have

$$|y(t)| \leqslant V(t, y(\cdot)) \leqslant V(t_0, \phi)\, e^{-b(t-t_0)} + \int_{t_0}^{t} e^{-b(t-s)} |f(s)|\, ds$$
$$\leqslant \|\phi\| [1 + (M/b)]\, e^{-b(t-t_0)} + M$$

for all $t \geqslant t_0$. This is uniform ultimate boundedness. With $|f(t)| \equiv 0$, the second M becomes zero, implying exponential asymptotic stability.

EXAMPLE 1. Consider the equation

$$y'(t) = (-1 + \sin t)\, y(t) + \int_0^t [my(s)/(1 + t - s)^2]\, ds + \cos t \tag{16}$$

with $m > 0$ and define

$$V(t, y(\cdot)) = |y| + \int_0^t \Big[-(\sin s)\, e^{-(t-s)}$$
$$- \int_s^t [me^{-(t-u)}/(1 + u - s)^2]\, du \,|y(s)|\, ds.$$

Then $\mu(t) + \alpha(t) = -1 = -b$ and (12) holds:

$$V(t, y(\cdot)) \leqslant V(t_0, \phi)\, e^{-(t-t_0)} + \int_{t_0}^{t} e^{-(t-s)} |\cos s|\, ds.$$

Now, define

$$-\Phi(t, s) = -(\sin s)\, e^{-(t-s)} - \int_s^t [me^{-(t-u)}/(1 + u - s)^2]\, du$$

when $\sin s \geqslant 0$ and

$$-\Phi(t,s) = -\int_s^t [me^{-(t-u)}/(1+u-s)^2]\,du$$

when $\sin s < 0$. Then

$$\int_0^t \Phi(t,s)\,ds = \int_0^t (\sin s)^+ e^{-(t-s)}\,ds$$

$$+ \int_0^t \int_s^t [me^{-(t-u)}/(1+u-s)^2]\,du\,ds,$$

where $(\sin s)^+ = \max[\sin s, 0]$. The first integral is bounded by some number $J < 1$. Also

$$\begin{aligned}\int_0^t \int_s^t & [me^{-(t-u)}/(1+u-s)^2]\,du\,ds \\ &= \int_0^t \int_0^u [me^{-(t-u)}/(1+u-s)^2]\,ds\,du \\ &= \int_0^t me^{-(t-u)}[1 - \{1/(1+u)\}]\,du \\ &< m.\end{aligned}$$

Thus, if $m + J \stackrel{\text{def}}{=} P < 1$ then (14) is satisfied so that the conditions of Corollaries 1.1 and 1.2 hold. In summary we have

Corollary 1.4. *If in* (16) *the constant m is small enough, then solutions of* (16) *are uniform bounded.*

Remark 2. The form of V may be generalized in a straightforward fashion. For if $g(t,s)$ is any continuous function and if $\hat{g}(t)$ is a continuous function satisfying

$$\mu(t) + \alpha(t) \leqslant -\hat{g}(t) \leqslant -g(t,s)$$

for $0 \leqslant s \leqslant t < \infty$, then

$$V(t, y(\cdot)) = |y| + \int_0^t \left[\alpha(s) \exp\left[-\int_s^t g(u,s)\,du\right] - \int_s^t \exp\left[-\int_u^t g(v,s)\,dv\right] |h(u,s)|\,du\right] |y(s)|\,ds$$

will yield

$$V'_{(2)}(t, y(\cdot)) \leqslant -\hat{g}(t)\, V(t, y(\cdot)) + |f(t)|$$

from which counterparts of the preceding results may be obtained.

Remark 3. In recent years several authors have given particularly simple criteria for solutions of (2) to converge to zero when $f(t) \equiv 0$ and $\mu(t)$ is a negative constant (cf. [2, 3, and 6]). Essentially either $\int_0^t |h(t, s)|\, ds \leqslant M < |\mu|$ or $\int_0^t |h(u, s)|\, du \leqslant M < |\mu|$ for $0 \leqslant s \leqslant t < \infty$ suffices. As (2) is linear, if solutions of (2) are bounded when $f(t) \not\equiv 0$, while solutions of (2) are asymptotically stable when $f(t) \equiv 0$, then all solutions of (2) will converge to a bounded and globally stable solution. The next corollary is a formal example of this.

COROLLARY 1.5. *Let* $\alpha(t) \equiv 0$, $\mu(t) = -b < 0$, *and let* $\int_0^u |h(u, s)|\, ds \leqslant \bar{b} < b$ *for* $0 \leqslant u < \infty$. *If there exists* $M > 0$ *with* $\int_0^t e^{-b(t-s)}\, |f(s)|\, ds \leqslant M$ *for* $0 \leqslant t < \infty$, *then* (2) *has a bounded solution which is globally asymptotically stable.*

Proof. By Remark 1 we see that (13) and (14) are satisfied. Thus, the conditions of Corollary 1.1 are satisfied and all solutions of (2) are bounded. Now $\int_0^u |h(u, s)|\, ds \leqslant \bar{b} < b$ is the condition needed in Theorem 5 of [6] when $f(t) \equiv 0$ and that theorem concludes that the solutions go to zero. The linearity of (2) and the superposition principle now complete the proof.

We now briefly indicate that these results generalize directly to (1) using the functionals in (6) and (7).

As all characteristic roots of A have negative real parts, there is a unique symmetric matrix B which is positive definite and which satisfies

$$A^T B + BA = -I. \tag{17}$$

As B is positive definite, there are constants r, k, and K (not unique) with

$$|x| \geqslant 2k(x^T Bx)^{1/2}, \tag{18}$$

$$|Bx| \leqslant K(x^T Bx)^{1/2}, \tag{19}$$

and

$$r\,|x| \leqslant (x^T Bx)^{1/2}. \tag{20}$$

Let M, p, and Q: $[0, \infty) \to [0, \infty)$ be continuous functions with properties

$$|x| \leqslant m \qquad \text{implies} \quad |E(x)| \leqslant M(m) \tag{21}$$

and

$$|x(s)| \leqslant q \qquad \text{for} \quad 0 \leqslant s \leqslant t \text{ implies } |F(t, x(\cdot))| \leqslant p(t)\, Q(q). \tag{22}$$

We define

$$V_2(t, x(\cdot)) = (x^T B x)^{1/2} + \int_0^t \left[\left\{ \alpha(s)\, e^{-b(t-s)} - \int_s^t K e^{-b(t-u)} \, |C(u, s)| \, du \right\} |E(x(s))\, x(s)| \right] ds$$

and compute the derivative of V_2 along a solution $x(t)$ of (1):

$$\begin{aligned}
&V'_{2(1)}(t, x(\cdot)) \\
&= \Bigg(\left\{ \left[x^T A^T + \int_0^t x^T(s)\, E^T(x(s))\, C^T(t, s)\, ds + F^T \right] Bx \right. \\
&\quad \left. + x^T B \left[Ax + \int_0^t C(t, s)\, E(x(s))\, x(s)\, ds + F \right] \right\} \Big/ 2(x^T B x)^{1/2} \Bigg) \\
&\quad + \alpha(t)\, |E(x) x| + \int_0^t \left[-b\alpha(s)\, e^{-b(t-s)} - K\, |C(t, s)| \right. \\
&\quad \left. + \int_s^t bK e^{-b(t-u)} \, |C(u, s)| \, du \right] |E(x(s))\, x(s)| \, ds \\
&\leqslant -k\, |x| + \int_0^t K\, |C(t, s)| \, |E(x(s))\, x(s)| \, ds \\
&\quad + K\, |F(t, x(\cdot))| + \alpha(t)\, |E(x)| \, |x| \\
&\quad - K \int_0^t |C(t, s)| \, |E(x(s))\, x(s)| \, ds \\
&\quad - b \int_0^t \left[\alpha(s)\, e^{-b(t-s)} - \int_s^t K e^{-b(t-u)} \, |C(u, s)| \, du \right] \\
&\quad \times |E(x(s))\, x(s)| \, ds \\
&\leqslant [-k + \alpha(t)\, |E(x)|] \, |x| + K\, |F(t, x(\cdot))| \\
&\quad - b \int_0^t \left[\alpha(s)\, e^{-b(t-s)} - \int_s^t K e^{-b(t-u)} \, |C(u, s)| \, du \right] \\
&\quad \times |E(x(s))\, x(s)| \, ds.
\end{aligned}$$

Now if there exists $m > 0$ and $b > 0$ such that

$$2k[-k + |\alpha(t)|\, M(m)] \leqslant -b,$$

so long as $|x(t)| \leqslant m$ we have

$$\begin{aligned} V'_{2(1)}(t, x(\cdot)) &\leqslant 2k(-k + \alpha(t)\, M(m))(x^T Bx)^{1/2} + Kp(t)\, Q(m) \\ &\quad - b \int_0^t \left[\alpha(s)\, e^{-b(t-s)} - \int_s^t Ke^{-b(t-u)}\, |C(u, s)|\, du\right] \\ &\quad \times |E(x(s))\, x(s)|\, ds \\ &\leqslant -bV_2(t, x(\cdot)) + Kp(t)\, Q(m). \end{aligned}$$

It is now possible to repeat Theorem 1 and its corollaries for (1) using V_2 and $V'_{2(1)}$. This computation is summarized as

THEOREM 2. *Let* (17)–(22) *hold and suppose there is a continuous function* α: $[0, \infty) \to R$ *together with positive constants b and m such that*

$$2k[-k + |\alpha(t)|\, M(m)] \leqslant -b.$$

If $x(t)$ *is a solution of* (1) *satisfying* $|x(t)| \leqslant m$ *on* $[0, \infty)$ *then*

$$V'_{2(1)}(t, x(\cdot)) \leqslant -bV(t, x(\cdot)) + KQ(m)\, p(t)$$

and, hence,

$$V_2(t, x(\cdot)) \leqslant V_2(t_0, \phi)\, e^{-b(t-t_0)} + KQ(m) \int_{t_0}^t e^{-b(t-s)} p(s)\, ds$$

for $t \geqslant t_0$.

COROLLARY 2.1. *Let the conditions of Theorem* 2 *hold and suppose there is a continuous scalar function* $\Phi(t, s) \geqslant 0$ *for* $0 \leqslant s \leqslant t < \infty$ *with*

$$\alpha(s)\, e^{-b(t-s)} - \int_s^t e^{-b(t-u)}\, |C(u, s)|\, du \geqslant -\Phi(t, s)$$

for $0 \leqslant s \leqslant t < \infty$ *and that*

$$\int_0^t \Phi(t, s)\, ds \leqslant P < r/M(m),$$

where r is defined in (20) *and P is a positive constant. If ϕ is any continuous initial function satisfying $|\phi(t)| < m$ on $[0, t_0]$ and if*

$$g(t, t_0, \phi) \overset{\text{def}}{=} V_2(t_0, \phi)\, e^{-b(t-t_0)} + \int_{t_0}^{t} KQ(m)\, e^{-b(t-s)} p(s)\, ds$$
$$< m[r - PM(m)]$$

for $t_0 \leqslant t < \infty$, then $x(t, \phi)$ is defined on $[t_0, \infty)$, $|x(t)| < m$, and if $|x(T)|$ is the maximum of $|x(t)|$ on $[0, T]$ for some $T > t_0$, then

$$|x(T)| \leqslant g(T, t_0, \phi)/[r - PM(m)].$$

Proof. As $|\phi(t)| < m$ on $[0, t_0]$ there is nothing to prove unless $|x(t)|$ has a maximum on $[0, T]$ at $t = T > t_0$. Thus, for $0 \leqslant t \leqslant T$ we have

$$\begin{aligned} &r\,|x(T)| - PM(m)\,|x(T)| \\ &\leqslant r\,|x(T)| - \int_0^T M(m)\, \Phi(T, s)\, |x(T)|\, ds \\ &\leqslant (x^T(T)\, Bx(T))^{1/2} - \int_0^T \Phi(T, s)\, |E(x(s))\, x(s)|\, ds \\ &\leqslant V_2(T, x(\cdot)) \leqslant g(T, t_0, \phi) < m[r - PM(m)] \end{aligned}$$

so that $|x(T)| < m$ on $[0, T]$ and, hence, $|x(t)| \leqslant m$ for all t. Then, in particular, we have

$$|x(T)| \leqslant g(T, t_0, \phi)/[r - PM(m)].$$

This completes the proof.

This corollary is the basic boundedness result. It is much simplified when $E(x) = I$, and it is simplified still further when $\alpha(s)$ is taken as zero. The interested reader should find no difficulty in producing counterparts of the other corollaries.

Remark 4. Example 1 showed how $\alpha(s)$ could be used in the scalar case when $\mu(t)$ was not strictly negative. The function V_4 defined in (8) is used in exactly the same way for (1) when A is not a constant matrix, but when $z' = A(t)z$ is strongly asymptotically stable. In that case the matrix $B = B(t)$. We outline the details as follows. The reader is invited to look closely at the discussion by Krasovskii [8; pp. 55–62] concerning the construction of a Liapunov function for $z' = A(t)z$.

For this part of the discussion take $E(x) = I$ and $F(t, x(\cdot)) = F(t)$. With B,

H, and R yet to be determined (but all of them symmetric and R scalar) we consider V_4 in (8) and compute $V'_{4(1)}$, use the Schwartz inequality, and find

$$
\begin{aligned}
&V'_{4(1)}(t, x(\cdot)) \\
&\qquad \leqslant x^T[A^T B + BA + B' + H]x + (1/2)\,|B(t)| \int_0^t |C(t,s)|\, ds\, x^T x \\
&\qquad + 2\,|F(t)|\,|B(t)|\,|x| \\
&\qquad + \int_0^t \{(1/2)\,|B(t)|\,|C(t,s)| + R(t,s)\}\, x^T(s)\, x(s)\, ds \\
&\qquad - b \int_0^t x^T(s) \left[H(s)\, e^{-b(t-s)} + \int_s^t R(u,s)\, e^{-b(t-u)}\, du \right] x(s)\, ds.
\end{aligned}
$$

It is then a matter of attempting to choose B, H, b, and R so that

$$V'_4 \leqslant -bV_4 + M\,|F|$$

for $|x|$ small.

3. A Measure of Attraction

We have demonstrated in Theorem 2 and its corollary the manner in which the nonlinearity $E(x)x$ and the functional $F(t, x(\cdot))$ are handled. It will greatly simplify matters, then, to take $E(x) = I$ and $F(t, x(\cdot)) = R(t)$ for $R\colon [0, \infty) \to R^n$ being continuous. The interested reader may readily supply the details in the more general case.

Examples show that if no sign properties are to be imposed on the kernel $h(t, s)$, then boundedness of solutions of the scalar equation

$$y' = -y + \int_0^t h(t,s)\, y(s)\, ds$$

require some condition such as

$$\int_0^t |h(u,s)|\, du < 1.$$

As progressively stronger stability properties are desired, one expects to strengthen the integral condition. This was touched on by Brauer [1] when he asked for

$$\int_0^t s\,|h(s)|\, ds \Big/ \int_0^t |h(s)|\, ds$$

to be sufficiently small in the convolution case in order to conclude uniform asymptotic stability.

As a motivation, consider the linear scalar ordinary differential equation

$$z' = \lambda(t)z + f(t)$$

with solutions

$$z(t) = z(t_0)\exp\left[\int_{t_0}^{t} \lambda(s)\,ds\right] + \int_{t_0}^{t} \exp\left[\int_{s}^{t} \lambda(u)\,du\right] f(s)\,ds.$$

If the homogeneous equation

$$z' = \lambda(t)z$$

is asymptotically stable, then the forced equation will have solutions ultimately bounded provided that

$$\left|\int_{t_0}^{t} \exp\left[\int_{s}^{t} \lambda(u)\,du\right] f(s)\,ds\right| \leqslant M$$

for some $M > 0$ and $t_0 \leqslant t < \infty$. (Solutions are *ultimately bounded* if there exists L such that any solution $x(t, \phi)$ satisfies $|x(t, \phi)| < L$ for sufficiently large t.) In other words the allowable size of f is intimately connected to λ. In a similar manner, the allowable size of F in (1) is connected to just how strongly $\int_0^t |C(u, s)|\,du$ converges.

In this section we focus our attention on the linear vector equation

$$x'(t) = Ax(t) + \int_0^t C(t, s)\,x(s)\,ds + R(t) \tag{23}$$

with (17)–(20) holding and R: $[0, \infty) \to R^n$ being continuous.

THEOREM 3. *Suppose there is a differentiable scalar function* λ: $[0, \infty) \to (0, \infty)$ *such that for* $0 \leqslant s \leqslant t < \infty$ *we have*

(i) $\int_t^\infty \lambda(u)\,|C(u, s)|\,du$ *existing and for some* $d > 0$, *with* $d \leqslant 1$, *then*

(ii)
$$2k\left(-k + K\int_t^\infty [\lambda(u)/\lambda(t)]\,|C(u, t)|\,du\right)$$
$$\leqslant -d\lambda'(t)/\lambda(t) \overset{\text{def}}{=} -\gamma(t) \leqslant 0.$$

Then the functional

$$V_1(t, x(\cdot)) = (x^T Bx)^{1/2} + \int_0^t \int_t^\infty K[\lambda(u)/\lambda(t)]\,|C(u, s)|\,du\,|x(s)|\,ds$$

satisfies

$$V'_{1(23)}(t, x(\cdot)) \leqslant -\gamma(t)\, V_1(t, x(\cdot)) + K\,|R(t)|$$

and, hence,

$$V_1(t, x(\cdot)) \leqslant V_1(t_0, \phi) \exp\left[-\int_{t_0}^{t} \gamma(s)\, ds\right] + \int_{t_0}^{t} K \exp\left[-\int_{s}^{t} \gamma(u)\, du\right] |R(s)|\, ds.$$

Proof. A calculation yields

$$\begin{aligned} V'_{1(23)}(t, x(\cdot)) &\leqslant -k\,|x| + K\,|R(t)| \\ &\quad + \int_t^{\infty} K[\lambda(u)/\lambda(t)]\,|C(u, t)|\, du\,|x| \\ &\quad - \gamma(t) \int_0^t \int_t^{\infty} K[\lambda(u)/\lambda(t)]\,|C(u, s)|\, du\,|x(s)|\, ds \\ &\leqslant -\gamma(t)\, V_1(t, x(\cdot)) + K\,|R(t)| \end{aligned}$$

from which the result follows.

The following example illustrates the simplicity, rather than the strength, of Theorem 3.

EXAMPLE 2. Let q: $[0, \infty) \to (-\infty, \infty)$ be continuous with $\int_{t_0}^{t} |q(s)|\, ds \leqslant M(t+1)^{1/2}$ for $0 \leqslant t_0 < \infty$ and some $M > 0$. If m is a sufficiently small positive constant, then every solution of

$$x'(t) = -x(t) + \int_0^t [m/(t-s+1)^3]\, x(s)\, ds + q(t)/(t+1)^{1/2}$$

satisfies $|x(t)| \leqslant (2M)^{1/2}$ if t is sufficiently large. (Notice that the forcing function need not be integrable, nor need it tend to zero.)

Proof. In terms of Theorem 3, $A = -1$, $B = 1/2$, $k = K = \sqrt{2}/2$, and we want to find $\lambda(t) > 0$ with (i) and (ii) holding.

Take $\lambda(t) = t + 1$ and have

$$\int_t^{\infty} [\lambda(u)/\lambda(t)]\,|C(u, t)|\, du = [m/(t+1)] \int_t^{\infty} [(u+1)/(u-t+1)^3]\, du.$$

Now, let $u - t + 1 = v$ so that this expression becomes

$$[m/(t+1)] \int_1^\infty [(v+t)/v^3] \, dv = [m/(t+1)][1 + (t/2)] \leqslant m.$$

Thus, in (ii) we have

$$2k^2 \left(-1 + \int_t^\infty [\lambda(u)/\lambda(t)] \, |C(u, t)| \, du \right) \leqslant -1 + m \leqslant -d/(t+1)$$

provided that $m \leqslant \frac{1}{2}$ and $d = \frac{1}{2}$. With these choices, $\gamma(t) = 1/2(t+1)$. Thus,

$$\begin{aligned} \exp - \int_s^t \gamma(u) \, du &= \exp - (1/2) \ln[(t+1)/(s+1)] \\ &= [(s+1)/(t+1)]^{1/2}. \end{aligned}$$

Then

$$\begin{aligned} &\int_{t_0}^t K \left[\exp - \int_s^t \gamma(u) \, du \right] |R(s)| \, ds \\ &\quad = [K/(t+1)^{1/2}] \int_{t_0}^t (s+1)^{1/2} \, |q(s)|/(s+1)^{1/2} \, ds \\ &\quad \leqslant [K/(t+1)^{1/2}] \, M(t+1)^{1/2} = KM < M. \end{aligned}$$

Thus, for large t we have $V(t, x(\cdot)) < M$ and, as $V(t, x(\cdot)) \geqslant x^2/2$, then $|x| \leqslant (2M)^{1/2}$ for large t. This completes the proof.

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